MTH 406 Midterm Solutions

1. (a) Show that a space curve γ is planar if, and only if, $\gamma' \cdot (\gamma'' \times \gamma''') = 0$.
   (b) Show that the curve $\gamma(t) = (a\cos(t), a\sin(t), \sin(t) + \cos(t) + b)$, where $a, b \in \mathbb{R}$ is a plane curve.

   Solution. (a) By Lesson Plan 1.5 (iv), we know that a curve is planar if, and only if, $\tau = 0$. Since

   $$\tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\dot{\gamma} \times \ddot{\gamma}\|},$$

   it follows that

   $$\tau = 0 \iff (\gamma' \times \gamma'') \cdot \gamma''' = 0.$$

   (b) This is left as an exercise, as it follows from a straightforward computation.

2. Consider the ellipse defined by

   $$\gamma(t) = (a\cos(t), b\sin(t)),\ t \in \mathbb{R} \text{ and } a, b > 0.$$

   Where is the curvature of γ maximal and minimal?

   Solution. We know from class that curvature is given by $\kappa(t) = \|\gamma''(t)\|$. Since

   $$\gamma'(t) = (a\cos(t), -b\sin(t)) \text{ and } \gamma''(t) = (-a\sin(t), -b\cos(t)),$$

   we have $\kappa(t) = a^2\sin^2(t) + b^2\cos^2(t)$. To determine the maximum and minimum values of $\kappa(t)$, we set

   $$\kappa'(t) = a^2\sin(2t) - b^2\sin(2t) = 0.$$

   This would imply that either $\gamma'(t) = 0$, in which case curvature is constant, or $\sin(2t) = 0$, which implies that $t = k\frac{\pi}{2}$, where $k \in \mathbb{Z}$. Further, we have that

   $$\kappa''(t)|_{\text{at } t=k\frac{\pi}{2}} = 2(a^2 - b^2)\cos(k\pi).$$

   Assuming WLOG that $a > b$, we see that $\kappa''(t) < 0$, when $k = 2\ell$, for $\ell \in \mathbb{Z}$, and thus on these values $\kappa(t)$ is maximum. By a similar reasoning, when $k = (2\ell + 1)$, for $\ell \in \mathbb{Z}$, the curvature is minimum.

3. Show that the sphere S^2 is a regular surface with a parametrization having exactly two coordinate neighborhoods.

Solution. Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ denote the north and south poles of S^2, respectively. The stereographic projections

$$\pi_N : S^2 \setminus \{N\} \to \mathbb{R}^2 : (x, y, z) \xmapsto{\pi_N} \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

and

$$\pi_S : S^2 \setminus \{S\} \to \mathbb{R}^2 : (x, y, z) \xmapsto{\pi_S} \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

are diffeomorphisms. Thus,

$$\pi_N^{-1} : \mathbb{R}^2 \to S^2 \setminus \{N\} \text{ and } \pi_S^{-1} : \mathbb{R}^2 \to S^2 \setminus \{S\}$$

together yield a parametrization of S^2 with coordinate neighborhoods $S^2 \setminus \{N\}$ and $S^2 \setminus \{S\}$.

4. Let $\gamma : (a, b) \to \mathbb{R}^3$ be a unit-speed curve whose curvature satisfies $0 < \kappa(t) < 1/\epsilon$, for all $t \in (a, b)$. Show that $\varphi : (a, b) \times (0, 2\pi) \to \mathbb{R}^3$ defined by

$$\varphi(t, \theta) = \gamma(t) + (\epsilon \cos(\theta))\eta(t) + (\epsilon \sin(\theta))b(t).$$

is a parametrized surface.

Solution. To show that φ is parametrized surface, we need to show φ is a differentiable function, that is, all partial derivatives of φ exist, and are continuous. Thus, we have to show that each partial derivative appearing in Jacobian matrix

$$\begin{bmatrix} \gamma'(t) & 0 \\ (\epsilon \cos(\theta))\eta'(t) & -(\epsilon \sin(\theta))\eta(t) \\ (\epsilon \sin(\theta))b'(t) & (\epsilon \cos(\theta))b(t) \end{bmatrix}$$

is well-defined and continuous. We know from class that $T(t) = \gamma'(t)$, $\eta(t)$ and $b(t)$ are continuous functions. Further, since the torsion $\tau(t)$ is a continuous function (why?), it follows by the Serret-Frenet equations that $b'(t)$ and $\eta'(t)$ are also continuous functions. Thus, each partial derivative appearing in the Jacobian is well-defined and continuous. Geometrically speaking, φ defines a surface that bounds a tubular neighborhood of γ whose cross-section is a circular disk of radius ϵ.

5. Let S be a regular surface.

   (a) Prove that is S is connected, then S is path-connected.

   (b) Prove that $U \subset S$ is a regular surface, if, and only if, U is open in S.

Solution. Since S is a regular surface, around each point $p \in S$, there exists a local coordinate neighborhood $V_p \cap S \ni p$ parametrized by $f_p : U_p(\subset \mathbb{R}^2) \to V_p \cap S$.

(a) By the standard topology in $\mathbb{R}^2$, there exists a (path-connected) open ball $B(f^{-1}(p), \epsilon_p) \subset U_p$. Since f_p is a homeomorphism, $W_p = f_p(B(f^{-1}(p), \epsilon_p))$ is a path-connected coordinate neighborhood of p. Hence S is locally path-connected, and since S is connected, it follows that S is path-connected (as a connected locally path-connected topological space is path-connected).

(b) Suppose that $U \subset S$ is open. Then, at each $p \in U$, there exists a local parametrization given by

$$f_p|_{f_p^{-1}(U)\cap U_p} : f_p^{-1}(U) \cap U_p \to U \cap (V_p \cap S).$$

Thus U is regular surface.

Conversely, assume that $U \subset S$ is regular surface. Then for each point $p \in U$, there exists local coordinates given by

$$f'_p : U'_p(\subset \mathbb{R}^2) \to V'_p \cap U.$$

Since V'_p is open in U, by the subspace topology, $V'_p = V''_p \cap U$, where V''_p is open in S. Thus, $U = \cup_{p\in U} V''_p \cap U$, which implies that U is open in S.

6. Determine whether the following pairs of surfaces are diffeomorphic.

   (a) $\{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2+y^2,\ z > 0\}$ and $\{(x, y, z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$.

   (b) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ and $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Solution. (a) Denote

$C = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2+y^2,\ z > 0\}$ and $A = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$.

Consider the map

$$\varphi : C \to A : (x, y, z) \overset{\varphi}{\mapsto} (x/z, y/z, \log(z)).$$

Since the three component functions of φ are smooth, it follows φ is a smooth map. Furthermore, we have

$$\varphi^{-1} : A \to C : (u, v, w) \xmapsto{\varphi^{-1}} (e^w u, e^w v, e^w),$$

which is also a smooth map, as its component maps are clearly smooth. Thus, φ is a diffeomorphism, and so C and A are diffeomorphic.

(b) Denote $D = \mathbb{R}^2 \setminus \{(0,0)\}$. We show that D and $\mathbb{R}^2 \setminus \{(0,0)\}$ are diffeomorphic, by establishing that D and A are diffeomorphic. Consider the restriction

$$\pi|_A : A \to D : (x, y, z) \overset{\phi}{\mapsto} (x, y)$$

of the projection map $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ onto the xy-plane. Clearly, this is a diffeomorphism, whose inverse is given by

$$\pi|_A^{-1} : D \to A : (u, v) \xmapsto{\pi|_A^{-1}} (u, v, (u^2 + v^2)^{\frac{1}{2}}).$$

7. **(Bonus)** Let γ be a unit-speed plane curve with nowhere-vanishing curvature. The *evolute* of γ is defined by

$$\epsilon(s) = \gamma(s) + \frac{n(s)}{\kappa_\pm(s)}.$$

Describe an infinite family of curves that have the same evolute.

Solution. Consider the circle $C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ parametrized by $\gamma(s) = (r\cos(s), r\sin(s))$, $s \in [0, 2\pi)$. Then, we have

$$T(s) = (-r\sin(s), r\cos(s)),$$

and so

$$n(s) = (-r\sin(s+\pi/2), r\cos(s+\pi/2)) = (-r\cos(s), -r\sin(s)) = -\gamma(s).$$

Thus, the evolute of the infinite family of concentric circles $\{C_r : r > 0\}$ is the degenerate point $(0, 0)$.